A Transformer Cascade, Optimally Matched to a Conductance Shunted by a Stub, Exhibiting Reflection Coefficient Zeros

H. J. RIBLET

***Abstract*—A class of transmission-line networks is exhibited, which consist of alternate quarter-wave line sections and short-circuited stubs terminated in a stub in parallel with a conductance, which are optimum in the sense that no other network of the same form, with the same termination, can have a lower bound on its reflection coefficient in the design band. The reflection coefficients of these optimum networks are equiripple and have the maximum number of zeros in the design band consistent with their length.**

The result of this letter is contained in the following theorem in which the Y_i are the characteristic admittances of the equal-length stubs of Fig. 1, and the Z_i are the characteristic impedances of transmission lines of the same length separating them, while R_0 is the terminating resistance and the generator impedance is unity.

Theorem

If, for $Y_i = 0$, $i = 1,\cdots,n$ and fixed Y_0 and R_0, the Z_i can be chosen so that the input reflection coefficient Γ has $n+1$ zeros in a design band and assumes its maximum absolute value, Γ_{max}, $n+2$ times in the design band, then there is no other network of the allowed form, terminated in Y_0 and R_0 for which $\Gamma \leq \Gamma_{max}$ in the design band.

Before proceeding with the proof of this theorem, it should be observed that Carlin and Kohler [1] have shown that for a fixed design band, and given $R_0 < 1$ or $Y_0 > 0$, a network can be found meeting the conditions of the theorem. Of course, R_0 and Y_0 cannot both be assigned arbitrarily. In fact, Carlin and Kohler have given formulas for determining R_0 and Y_0 in terms of Γ_{max}.

Proof: If $t = \cos\theta/\sin\theta$, with $\theta = 2\pi l/\lambda_g$, the transfer matrix of the matching network, including the last susceptance, may be written

$$(t^2+1)^{-n/2}\begin{pmatrix} 1 & 0 \\ -jY_nt & 1\end{pmatrix}\begin{pmatrix} t & jZ_n \\ j/Z_n & t\end{pmatrix}\begin{pmatrix} 1 & 0 \\ jY_{n-1}t & 1\end{pmatrix}\cdots \begin{pmatrix} 1 & 0 \\ -jY_1t & 1\end{pmatrix}\begin{pmatrix} t & jZ_1 \\ j/Z_1 & t\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -jY_0t & 1\end{pmatrix} \quad (1)$$

or

$$(t^2+1)^{-n/2}\begin{pmatrix} t & jZ_n \\ -j(Y_nt^2-1/Z_n) & (Y_nZ_n+1)t\end{pmatrix}\cdots \begin{pmatrix} t & jZ_1 \\ -j(Y_1t^2-1/Z_1) & (Y_1Z_1+1)t\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -jY_0t & 1\end{pmatrix} \quad (2)$$

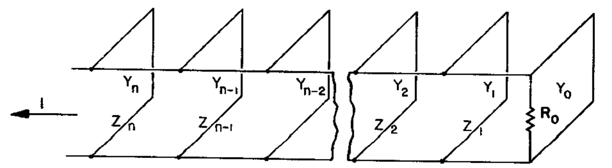


Fig. 1.

Manuscript received May 7, 1975; revised August 15, 1975.
The author is with the Microwave Development Laboratries, Inc., Needham Heights, MA 02194.

which is readily shown by induction to have the general form

$$(t^2+1)^{-n/2}\begin{pmatrix} A_n(t) & jB_{n-1}(t) \\ jC_{n+1}(t) & D_n(t)\end{pmatrix}. \quad (3)$$

Here $A_n(t)$, $B_{n-1}(t)$, $C_{n+1}(t)$, and $D_n(t)$ are real, even or odd polynominals in t depending on the degree given by their subscript. Finally the insertion loss function, P_L, is given by

$$P_L = 1 + \frac{[R_0A_n(t) - D_n(t)]^2 + [R_0C_{n+1}(t) - B_{n-1}(t)]^2}{4R_0(t^2+1)^n} = 1 + \frac{E_{n+1}(t^2)}{(t^2+1)^n}. \quad (4)$$

Now the result is that in minimizing this function over the design band, we are dealing with what amounts to a Chebyshev problem [2]. Thus we are especially interested in the coefficient of the highest power of t in $E_{n+1}(t^2)$ and we see that it is determined by the coefficient of t^{n+1} in $C_{n+1}(t)$. Consider

$$\begin{pmatrix} a_{i,0}t^i + \cdots & j[b_{i,0}t^{i-1} + \cdots] \\ j[c_{i,0}t^{i+1} + \cdots] & d_{i,0}t^i + \cdots\end{pmatrix} = \begin{pmatrix} t & jZ_i \\ -j(Y_it^2 - 1/Z_i) & (Y_iZ_i+1)t\end{pmatrix} \cdot \begin{pmatrix} a_{i-1,0}t^{i-1} + \cdots & j[b_{i-1,0}t^{i-2} + \cdots] \\ j[c_{i-1,0}t^i + \cdots] & d_{i-1,0}t^{i-1} + \cdots\end{pmatrix}. \quad (5)$$

Then

$$a_{i,0} = a_{i-1,0} - Z_ic_{i-1,0} \quad (6)$$

and

$$c_{i,0} = -Y_ia_{i-1,0} + (Y_iZ_i+1)c_{i-1,0} \quad (7)$$

while $a_{0,0} = 1$ and $c_{0,0} = -Y_0$. Since Z_i is always positive and Y_i is never negative, $a_{i,0} > 0$ while $c_{i,0} \leq c_{i-1,0}$ with the equality holding if and only if $Y_i = 0$. Thus the coefficient of t^{n+1} in $C_{n+1}(t)$ is less than $-Y_0$ unless all of the Y_i are zero in which case it is equal to $-Y_0$, regardless of the values of the Z_i.

For the particular Z_i which were selected so that the conditions of the theorem are satisfied

$$E_{n+1}(t^2)/(t^2+1)^n$$

has the appearance in the design band shown in Fig. 2 Clearly any other function of the form

$$E_{n+1}'(t^2)/(t^2+1)^n$$

arising from an allowed network with the same load must have a leading coefficient C' for which $C' \geq Y_0^2R_0/4$. If we assume that it has equal or superior performance in the design band so that

$$E_{n+1}'(t^2)/(t^2+1)^n \leq \Gamma_{max}^2/(1+\Gamma_{max}^2) \quad (8)$$

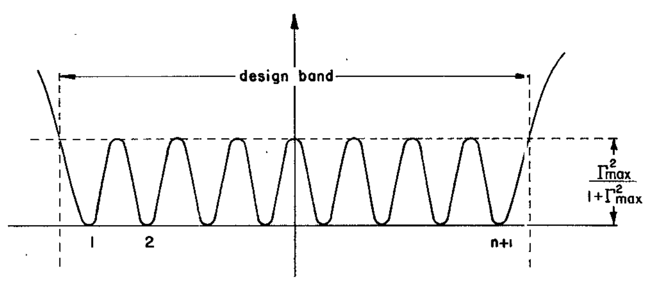


Fig. 2.

it will still have values falling between these limits when multiplied by $Y_0^2R_0/4C'$. The new function will intersect the curve in Fig. 2 in $2n + 2$ points, and the difference between the new function and the equiripple function defines an even polynomial of degree $2n$ with $2n + 2$ zeros. Since this is clearly impossible unless $E_{n+1}'(t^2) = E_{n+1}(t^2)$, the conditions of the theorem characterize a unique, optimum transformer cascade which matches the specified reactively shunted load with equiripple performance and reflection coefficient zeros in the design band.

COMMENTS

One of the principal points of interest in this letter arises from the fact that this example shows that an integral condition on the logarithm of the absolute value of the reflection coefficient like [1, eq. (58)] cannot be used to argue that in an optimum match to a reactively loaded load, one must avoid reflection coefficient zeros in the design band. The situation here for distributed matching networks should be compared to that for lumped constant networks discussed extensively by Fano [3].

The argument used in the proof of the theorem is very similar to that which one would use to prove that the Chebyshev polynomials provide the solution to the problem of approximating $2^{n-1}x^n$ as closely as possible to a polynomial of lower degree, a so-called Chebyshev problem [2]. In this case, however, the coefficient of the even function $t^{2n+2}/(1 + t^2)^n$, which is to be approximated, is not fixed by the class of networks to be considered, with the consequence that we are not dealing with a true Chebyshev problem. Nevertheless, since all the networks of the permitted class result in a larger, leading coefficient than is associated with the network which provides the behavior required by the Chebyshev criterion, a proof that it is optimum can still be constructed. It may be of future interest to refer to such a problem as quasi-Chebyshev problem.

It should also be noted that similar remarks apply to the dual network in which the termination consists of a resistance in series with a capacitive stub.

REFERENCES

[1] H. J. Carlin and W. Kohler, "Direct synthesis of band-pass transmission line structures," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 283–297, May, 1965.
[2] N. I. Achieser, *Theory of Approximation.* New York: Ungar, 1956, pp. 51–58.
[3] R. Fano, "Theoretical limitations on the broadband matching of arbitrary impedances," *J. Franklin Inst.*, vol. 249, pp. 57–83, 139–154, Jan.–June, 1950.

Contributors

Colin D. Corbey was born in Amersham, England, on May 31, 1944. He received the B.Sc. degree in electrical engineering from the University College of Swansea, Swansea, Wales, in 1965, and the diploma in microwave engineering and the M.Sc. degree from the University College of London, London, England, in 1966.

He joined the Mullard Research Laboratories, Redhill, Surrey, England, in 1966, where he has been engaged in research and development of varactor-diode microwave harmonic generators and broad-band varactor-tuned oscillators using Gunn and IMPATT diodes. At present he is working on high-power pulsed microwave sources.

★

Robert Davies was born in Cardiff, Wales, on September 15, 1938. He received the B.Sc. degree in electrical engineering from the University College of South Wales and Monmouthshire, Cardiff, in 1959, and the Ph.D. degree from Queens University of Belfast, Belfast, North Ireland, in 1964.

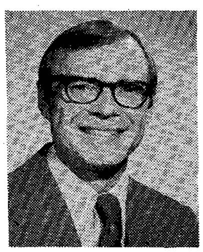
He joined Mullard Research Laboratories, Redhill, Surrey, England, in 1963, and has since been engaged in the characterization and application of microwave semiconductor devices. He now leads a group concerned with the use of microwave semiconductor devices in circuits and subassemblies involving microwave integrated circuit (MIC) parametric amplifiers, transistor and FET low-noise amplifiers, transistor power amplifiers and frequency multipliers, TRAPATT oscillators, frequency-stable sources, and ferrite-diode limiters.

Pietro de Santis (M'65) was born in Rome, Italy, on November 24, 1937. He received the "Dottore in Ingegneria Elettronica" degree with highest honors from the University of Rome, Rome, Italy, in 1962 and the M.S. degree in electrophysics from the Polytechnic Institute of Brooklyn, New York, N.Y., in 1965.

In 1962 he joined the Research Department of Selenia S.p.A., Rome, where he was engaged in research work on microwave plasmas and ferrites. In 1971 he received the Libera Docenza in electromagnetic fields and circuits. From 1969 to June 1975 he was Professore Incaricato of microwaves at the University of Naples, Naples, Italy. He is the Italian Representative in the management committee of the European Microwave Conference.

Dr. de Santis is a member of the American Physical Society, Associazione Elettrotecnica Italiana, and a corresponding member of the SMAG/TC HFM.

★



Carl H. Durney (S'60–M'64) was born in Blackfoot, ID, on April 22, 1931. He received the B.S. degree in electrical engineering from Utah State University, Logan, in 1958, and the M.S. and Ph.D. degrees in electrical engineering from the University of Utah, Salt Lake City, in 1961 and 1964, respectively.

From 1958 to 1959 he was employed as an Associate Research Engineer with the Boeing Airplane Company, Seattle, WA, where he studied the use of delay lines in control systems. He has been with the University of Utah since 1963, when he was appointed an As-